1. A die is manipulated so that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the dir, find $P(E)$. (2p)

   Solution: $P(E) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$.

2. The total number of hours (measured in units of 10^2 hours), that a family runs a vacuum cleaner over a period of one year is a continuous random variable X with probability density function

$$f(x) = \begin{cases} x\,, & \text{if } 0 < x < 1 \\ 2 - x\,, & \text{if } 1 \le x < 2 \\ 0\,, & \text{elsewhere} \end{cases}$$

   (a) Find the cumulative distribution function for X. (2p)

   (b) Find the expected value of X. (1p)

3. The interval $[2.2, b]$ is a 99% confidence interval for σ^2, computed from an observed random sample $x_1, \ldots, x_{30}$ coming from a normal distribution $N(\mu, \sigma)$. Compute the sample variance s^2 and the upper bound b. (3p)

   Solution:

$$s^2 = \frac{2.2}{29} \cdot \chi^2_{0.005} \approx 3.47.$$

$$b = \frac{29s^2}{\chi^2_{0.995}} \approx 8.8$$

4. The lifetime of a certain type of small motor is normally distributed with average life of 10 years with a standard deviation of 2 years. The manufacturer replaces wihtout charge all motors that fail while under guarantee.

   (a) If he is willing to replace only 3% of the motors that fail, how long should the guarantee time be? (2p)

   (b) If a customer buys 7 motors, what is the probability that at least 2 of these will fail before 7 years of use? (2p)

   Solution: (a) If T is the guarantee time, then we want to find T such that

$$P(X < T) = 0.03 \Leftrightarrow \Phi\left(\frac{T-10}{2}\right) = 0.03 \Leftrightarrow \frac{T-10}{2} \approx -1.88$$

   which leads to $T \approx 6.24$ years.

   (b) If Y is the number of engines that fails before 7 years of use, then $Y \sim \text{Bin}(7, p)$ where

$$p = P(X < 7) = \Phi\left(\frac{7-10}{2}\right) = \Phi(-1.5) \approx 0.067.$$

   Therefore,

$$P(Y \ge 2) = 1 - P(Y = 0) - P(Y = 1) = 1 - (1-p)^7 - 7p(1-p)^6 \approx 0.075$$

5. The numbers

$$2.0 \quad 3.2 \quad 3.8 \quad 2.5 \quad 3.3 \quad 2.8 \quad 3.0 \quad 3.4$$

   are observed values of a random sample from a normal distribution $N(\mu, \sigma)$.

   (a) Find a 95% confidence interval for μ if $\sigma = 0.6$. (1p)

   (b) Find a 95% confidence interval for μ if σ is unknown. (1p)

   (c) If σ is unknown, perform a two sided hypothesis test of $\mu = 3.2$ vs. $\mu \neq 3.2$ at significance level 5%. Also, compute the P-value for this test. (2p)

6. The weight of a certain crab is known to have an expected value of 1.4 kg and standard deviation of 0.4 kg. The distribution is unknown. One net contains 112 crabs.

(a) What is the probability that the total weight of the crabs is less than 150 kg? (2p)

(b) Find the weight w such that the probability of 112 crabs having a total weight larger than w is 5%. (1p)

7. It is claimed that a student can increase his score on a certain test by at least 50 points if he is provided with sample problems in advance. To test this claim, 20 students are divided into 10 pairs such that each matched pair has almost the same grades. Sample problems and answers are provided to one member of each paior one week prior to the examination. Then the following examination scores were recorded: (3p)

Pair	1	2	3	4	5	6	7	8	9	10
With sample problems	531	621	663	579	451	660	591	719	543	575
Without sample problems	513	540	688	502	424	683	568	748	530	524

Perform a Wilcoxon signed-rank test at significance level 5% in order to test the hypothesis that acess to the sample problems increase the score by 50 points against the alternative that the increase is less than 50 points.

Solution: Let μ_1 and μ_2 be the mean score of all students taking the test with and without sample problems, repectively. With $\mu_D = \mu_1 - \mu_2$ we want to test

$$\begin{aligned} H_0 &: \quad \mu_D = 50 \\ H_1 &: \quad \mu_D < 50 \end{aligned}$$

at significance level $\alpha = 0.05$.

Pair	1	2	3	4	5	6	7	8	9	10
With sample problems	531	621	663	579	451	660	591	719	543	575
Without sample problems	513	540	688	502	424	683	568	748	530	524
d_i	18	81	-25	77	27	-23	23	-29	13	51
$d_i - 50$	-32	31	-75	27	-23	-73	-27	-79	-37	1
Signed ranks	-6	5	-9	3.5	-2	-8	-3.5	-10	-7	1

The thest statistic is $w^+ = 1 + 3.5 + 5 = 9.5$. By the table we find the critical value $w^*_{0.05} = 10$. Since $w^+ \leq 10$ we reject H_0 at 5% significance - the sample problems do not, on the average, increase the score by as much as 50 points.

8. Three cards are drawn in succession, without replacement from an ordinary deck of playing cards. Find the probability that the event $E_1 \cap E_2 \cap E_3$ occurs, where E_1 is the event that the first card is a red ace, E_2 is the event that the second card is a 10 or a jack, and E_3 is the event that the third card is greater than 3 but less than 7. (3p)

Solution:

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3) &= P\left((E_1 \cap E_2) \cap E_3\right) \\ &= P(E_1 \cap E_2)P(E_3|E_1 \cap E_2) \\ &= P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2) \\ &= \frac{2}{52} \cdot \frac{8}{51} \cdot \frac{12}{50} = \frac{8}{5525} \end{aligned}$$